FREE ACTIONS ON $\mathbb{Z}^n$-TREES: A SURVEY

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Abstract. We survey the topic of free isometric actions of groups on $\mathbb{Z}^n$-trees. We record some new results not explicitly stated in the literature.

1. Introduction

The theory of isometric actions of groups on Λ-trees (where Λ is a linearly ordered abelian group) has its origins in a paper of Lyndon where integer-valued functions ('length functions') on groups satisfying two particular sets of axioms are studied, and it is shown that a group that admits such a function is a free group in one case and a free product in the other case. Chiswell later considered a weaker set of axioms for length functions and demonstrated a duality between groups equipped with such a length function and group actions by graph automorphisms on trees. The sets of axioms considered by Lyndon then correspond to the case where the vertices have trivial stabilisers (in the first case) and the edges have trivial stabilisers (in the second case).

By the time this result was published, another fundamental duality had been discovered, namely Bass-Serre theory, which gives an equivalence between group actions (without inversions) on a tree and so-called graphs of groups. This theory has become an important and well-established tool in group theory, exhibiting how a group that admits a suitably non-trivial action on a tree is built up from its vertex stabilisers (and the fundamental group of the corresponding quotient graph) by amalgamated free products and HNN extensions.

Group actions on trees are thus a well-established theme in combinatorial group theory, and the dualities outlined above give two ways to think, in particular, of free groups; namely as a group equipped with a length function, and as a group acting on a tree with trivial vertex stabilisers. It is interesting to note that these viewpoints give a geometric understanding of the combinatorial arguments, attributed to Nielsen and Schreier respectively, of the classical theorem that bears their names. Group actions on trees thus provide a way of unifying, and making more natural, the original proofs of this theorem. (Length functions and Bass-Serre theory similarly give two new proofs of Kuroš's theorem concerning subgroups of free products.)

Lyndon's arguments make relatively little use of properties peculiar to the integers, prompting him to speculate that the study of integer-valued length functions on groups might be profitably generalised to that of length functions whose codomain is the real additive group or, more generally, an ordered abelian group. The definition of Λ-tree where Λ is an ordered abelian group was given some 22 years later, by Morgan and Shalen, in the course of their work on compactifications of character varieties. One is led to the notion of an isometric group action on a Λ-tree, and thus to that of a Λ-valued length function, just as in the case of action of an ordinary tree: given a fixed basepoint u, one defines $L(g) = L_u(g) = d(u, gu)$ for each group element g. One can axiomatise the Λ-valued functions that arise in this way, giving an analogue of Chiswell's theorem for an arbitrary ordered abelian group Λ. That is, there is a natural duality between Λ-valued functions on a group G satisfying a certain set of axioms on the one hand, and isometric actions of G on Λ-trees on the other.

The natural challenge, posed by Alperin and Bass in their detailed study of isometric actions on Λ-trees, is to develop a systematic theory of such actions akin to Bass-Serre theory, which gives as complete a description as one could hope for of groups from their actions on Λ-trees in the special case $\Lambda = \mathbb{Z}$. Since it is now generally considered too ambitious to attempt to produce a single theory for arbitrary Λ, the more modest goal is to look at particular classes of ordered abelian groups, and to look at certain classes of actions. In particular, free actions of groups on $\mathbb{R}$-trees have received a great deal of attention from geometric group theorists. It was conjectured by Lyndon (in the language of length functions) that a group that admits a free action on an $\mathbb{R}$-tree is a free product of subgroups of $\mathbb{R}$. The first counterexamples were described by Alperin and Moss in 1980, and the first finitely generated counterexamples were given by Morgan and Shalen in 1991, when they showed that the fundamental groups of closed surfaces (with three exceptions) also admit free isometric actions on $\mathbb{R}$-trees. Rips later showed that these are the only freely indecomposable finitely generated counterexamples.

Theorem 1 (Rips' Theorem)**.** *Let G be a finitely generated group acting freely on an $\mathbb{R}$-tree. Then G is the free product of free abelian and surface groups, where any surface is permitted except the connected sum of one, two or three projective planes.*

Although group actions on $\mathbb{R}$-trees have received much attention, particularly in the last 12 years or so, and are now relatively well understood, there has been somewhat less progress in the study of group actions on Λ-trees for more exotic Λ. There is perhaps a perception that consideration of group actions on such Λ-trees is more trouble than it is worth. It is our contention, however, that such group actions should be more widely known, particularly in the case where

$$\Lambda = \mathbb{Z}^n = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n}$$

for some $n \in \mathbb{N}$. This is due to a combination of factors: firstly there are now several interesting examples of such actions arising 'in nature'; secondly, there has been much progress in the understanding of group actions on $\mathbb{Z}^n$-trees, as a result of work by Bass; and thirdly, there are a number of very interesting open questions to guide further research.

Readers wishing to learn about group actions on $\mathbb{R}$-trees have a variety of good surveys to choose from, notably [4], [21] and [9]. For the case of general Λ, we refer the reader to [17] where Λ-trees are first defined, [1] for the first systematic (and technical) treatment and [7] for an accessible and uptodate account.

In the course of this article we shall review the general theory of Λ-trees and then look more closely at free isometric actions on $\mathbb{Z}^n$-trees. It should be noted that this last topic has received some interest of late due in connections with the solution of the Tarski problem in [11], [12] and also in [20].

2. Basic theory of Λ-trees

For this section, Λ will denote an arbitrary ordered abelian group, which will be written additively. That is to say, $(\Lambda, +)$ is an abelian group, and $\leq$ is a linear (i.e. total) order on Λ satisfying $a \leq b \Rightarrow a + c \leq b + c$.

Let Λ_i $(i = 1, 2, \ldots, n)$ be ordered abelian groups. We define a linear order on $\prod \Lambda_i = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$, by declaring $(\lambda_i)_{i=1}^n < (\lambda'_i)_{i=1}^n$ if $\lambda_{i_0} < \lambda'_{i_0}$ where $i_0 = \min\{i : \lambda_i \neq \lambda'_i\}$. This is called the *lexicographic order* on $\prod \Lambda_i$. It should be noted that this order depends very much on the direct decomposition of Λ: there are countably infinitely many ways of defining such an order on $\mathbb{Z} \times \mathbb{Z}$, depending on which basis one chooses. Nor are lexicographic orders the only natural orders on $\mathbb{Z} \times \mathbb{Z}$, for the subgroups $\langle 1, \alpha \rangle$ of the real additive group give uncountably many distinct linear orders on the free abelian group of rank 2, each compatible with the group structure.

It is clear that every subgroup of an ordered abelian group is naturally an ordered abelian group in its own right. Let Λ' be a subgroup of Λ with the property that if $a, c \in \Lambda'$ and $a \leq b \leq c$ then $b \in \Lambda'$. We then call Λ' a *convex* *subgroup of* Λ. Examples include the trivial subgroup 0, and Λ itself. In the case where $\Lambda \leq \mathbb{R}$ these are the only examples. Conversely, if Λ has at most two convex subgroups then there is an order-preserving embedding of Λ in $\mathbb{R}$. Such an ordered abelian group is said to be *archimedean*. Archimedean ordered abelian groups are also characterised by the property that for all $x, y \in \Lambda$ with $y \neq 0$, there exists $n \in \mathbb{Z}$ with $x < ny$.

The ordered abelian group $\prod \Lambda_i$ has convex subgroups of the form $0 \times \cdots \times 0 \times \Lambda_{i+1} \times \cdots \times \Lambda_n$. If each Λ_i is archimedean, these are the only convex subgroups. If Λ is finitely generated, then Λ is expressible as a direct product of archimedean Λ_i with the lexicographic order.

A Λ-*metric space* is a pair (X, d), where $d : X \times X \longrightarrow \Lambda$ is function satisfying

M1. $d(x, y) = 0 \Leftrightarrow x = y$; and

M2. $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

If $\Lambda = \mathbb{R}$ this is equivalent to the usual definition of metric space. Note that in general, we have the additional familiar properties that d takes non-negative values and is symmetric. The most obvious generic example of a Λ-metric space is Λ itself, with $d(\lambda, \mu) = |\lambda - \mu| = \max\{\lambda - \mu, \mu - \lambda\}$.

An *isometry* between Λ-metric spaces is a function that preserves distances. Note that while such a function is guaranteed to be injective (by M1), it need not be surjective. A *segment in* Λ is a set of the form

$$[\lambda, \mu]_\Lambda = [\mu, \lambda]_\Lambda = \{\kappa \in \Lambda : \lambda \leq \kappa \leq \mu\}$$

where $\lambda \leq \mu$. (The subscript will of course usually be suppressed.) Let X be a Λ-metric space. A *segment in* X is an isometry $\phi : [0, \alpha]_\Lambda \longrightarrow X$. The points $\phi(0)$ and $\phi(\alpha)$ are called the endpoints of the segment ϕ. If every pair of points of X arise as the endpoints of some segment, then X is said to be *geodesic*. If, for each x and y in X, there is a unique segment $\phi : [0, \alpha] \longrightarrow X$ with $\phi(0) = x$ and $\phi(\alpha) = y$, then X is said to be *geodesically convex*, and the image of such a segment in X is denoted $[x, y] = [x, y]_X$.

We can now give a definition of Λ-tree: this is a Λ-metric space (X, d) which is geodesically convex and which satisfies the following properties, for all $x, y, z \in X$.

(i) $[x, y] \cap [y, z] = [y, w]$ for some $w \in X$;

(ii) $[x, y] \cap [y, z] = \{y\} \Rightarrow [x, z] = [x, y] \cup [y, z]$.

The reader should convince herself that in the case $\Lambda = \mathbb{Z}$, this definition describes those metric spaces that arise as the path metric on the vertex set of a tree T in the ordinary sense. Moreover, there is a natural one-to-one correspondence between graph automorphisms of T and isometries of $V(T)$. The study of group actions by automorphisms on trees is thus equivalent to the study of isometric actions on $\mathbb{Z}$-trees.

Each ordered abelian group Λ is a Λ-tree considered as a Λ-metric space. Since we will be primarily concerned with $\mathbb{Z}^n$-trees, let us give a concrete description of a simple $(\mathbb{Z} \times \mathbb{Z})$-tree.

Let X denote the $(\mathbb{Z} \times \mathbb{Z})$-tree $\{(j, n) : n \in \mathbb{Z},\ j = 0, 1\}$, where

$$d((j_1, n_1), (j_2, n_2)) = \begin{cases} (0, |n_1 - n_2|) & \text{if } j_1 = j_2 \\ (1, n_2 - n_1) & \text{otherwise.} \end{cases}$$

This is perhaps the simplest example of a $(\mathbb{Z} \times \mathbb{Z})$-tree, and is best pictured as a pair of copies of $\mathbb{Z}$ joined end to end. Here we have two 'balls of radius $0 \times \mathbb{Z}$', namely the sets $\{(j, n) : n \in \mathbb{Z}\}$ $(j = 0, 1)$, and an associated 'quotient' $\mathbb{Z}$-tree consisting of two vertices. Note however, this is not the only compatible $(\mathbb{Z} \times \mathbb{Z})$-metric that can be defined on X: for $k \in \mathbb{Z}$, we have the metric d_k given by $d_k((0, n), (1, m)) = (1, m - n + k)$.

More generally, a $(\mathbb{Z} \times \Lambda_0)$-tree is best pictured as a $\mathbb{Z}$-tree, where each vertex is replaced by a Λ_0-tree, and for each edge incident to a particular vertex, there is an associated 'end' or 'point at infinity' of the Λ_0-tree. As the multitude of compatible metrics on X above suggests, a $(\mathbb{Z} \times \Lambda_0)$-metric is not determined by these Λ_0-trees together with the $\mathbb{Z}$-metric on the set of Λ_0-trees: the metric is determined in addition by a choice of 'end maps' for each adjacent pair of Λ_0-trees.

There is a classification of (bijective) isometries $\sigma : X \longrightarrow X$ of Λ-trees analogous to that of ordinary trees. Either (i) σ fixes a point, in which case the set of all fixed points forms a *subtree* — that is, a subset A such that $x, y \in A$ implies $[x, y] \subseteq A$; (ii) σ fixes no point and there is a σ-invariant segment $[x, y]$ which is fixed by σ^2; or (iii) there is a σ-invariant linear subtree A_σ of X on which σ acts by translations. Moreover, A_σ contains every linear σ-invariant subtree in this case. These are called elliptic isometries, inversions and hyperbolic isometries respectively and the set A_σ is called the characteristic set of σ. It is convenient to let A_σ denote the set of fixed points when σ is elliptic, and to put $A_\sigma = \emptyset$ when σ is an inversion.

Definition 2. *An isometric action of a group on a Λ-tree X is* free *if there are no inversions and the stabiliser of each point of X is trivial. We say that a group G is Λ-free if G admits such an action on some Λ-tree, and say that G is* tree-free *if it is Λ'-free for some Λ'.*

2.1. Properties of tree-free groups

Theorem 2.1.

(i) Λ*, as an additive group, is Λ-free. In particular, the natural action of Λ on itself by translations is free.*

(ii) *The class of $\mathbb{Z}$-free groups is precisely the class of free groups.*

(iii) *The class of Λ-free groups is closed under taking subgroups.*
(iv) *If G is Λ-free and Λ embeds (as an ordered abelian group) in Λ', then G is Λ'-free.*
(v) *Any tree-free group is torsion-free.*
(vi) *Tree-free groups have the CSA property. That is, every maximal abelian subgroup, M, is malnormal; meaning that $M^g \cap M = 1$ for all $g \notin M$.*
(vii) *Commutativity is a transitive relation on the non-identity elements of a tree-free group.*
(viii) *Soluble subgroups of tree-free groups are abelian.*
(ix) *If G is Λ-free, then any abelian subgroup can be embedded in Λ. If in addition G is finitely generated, G is free abelian.*
(x) *Tree-free groups cannot contain Baumslag-Solitar groups other than $\mathbb{Z} \times \mathbb{Z}$. That is, no group of the form $\langle a, t : t^{-1}a^p t = a^q \rangle$ can be a subgroup of a tree-free group unless $p = q = \pm 1$.*
(xi) *Any two generator subgroup of a tree-free group is either free or free abelian.*
(xii) *The class of tree-free groups is closed under taking free products.*

We note that (i) is elementary and we have already observed that group actions on $\mathbb{Z}$-trees are equivalent to group actions on simplicial trees. Hence (ii) is the well known statement that free groups are the groups which act freely on trees. Property (iii) is elementary, (iv) is a manifestation of the base change functor – see Theorem 2.3 – and (v) is a consequence of the classification of isometries. Property (vi) holds because two non-trivial elements of a Λ-free group G commute if and only if they act as translations along the same linear subtree. It follows that proper centralisers of G, maximal abelian subgroups of G and stabilisers of axes of non-trivial elements of G describe the same subgroups of G. Thus if $\gamma \neq 1$ belonged to a maximal abelian subgroup $M = C_G(x)$ as well as to $M^g = C_G(x^g)$ it would follow that $A_x = A_\gamma = A_{x^g}$, whence $M = M^g$. A simple geometric argument shows that g cannot stabilise a linear subtree unless it is contained in A_g. Thus $A_g = A_x$, giving $g \in M$.

Properties (vii) and (viii) can now be easily deduced from (vi). Property (ix) relies on the same reasoning as in (vi), that an abelian subgroup must stabilise and act as translations along a Λ-line and a similar argument may be used to prove (x). The verification of (xi) requires a more involved analysis, which we shall not attempt here but can be found in [7].

To give some indication of why free products of Λ-free groups are Λ-free, we introduce the notion of a Λ-*valued Lyndon length function* on a group G. This is a function $L : G \longrightarrow \Lambda$ satisfying axioms L1-L4 below. First

define $c(g,h)$ for a pair of elements of G to be $\frac{1}{2}(L(g)+L(h)-L(g^{-1}h))$, noting that this is an element of $\frac{1}{2}\Lambda \supset \Lambda$.
L1: $c(g,h) \in \Lambda$
L2: L(1)=0
L3: $L(g^{-1}) = L(g)$
L4: $c(g,h) > c(g,k) \Rightarrow c(g,k) = c(h,k)$

Examples of Λ-valued Lyndon length functions include the following. Let (X,d) be a Λ-tree and $x \in X$, and suppose that G acts isometrically on X. Then putting $L(g) = L_x(g) = d(x,gx)$ for $g \in G$ defines a Lyndon length function. This in fact accounts for all examples.

Theorem 2.2 (Chiswell)**.** *Let L be a Λ-valued Lyndon length function on a group G. There exist a Λ-tree (X,d), a point $x \in X$, and an isometric action of G on X such that $L(g) = d(x,gx)$ for all $g \in G$.*

In fact Chiswell proved a somewhat stronger result concerning the uniqueness of such an X — we refer the reader to his book [7] for details.

The Lyndon length function associated with an isometric action on a Λ-tree is not the only one: one also has the associated *translation length function* , given by

$$\|g\| = \begin{cases} \min\{d(x,gx) : x \in X\} & \text{if } g \text{ is not an inversion} \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that this minimum is always realised. We record two properties relating the two length functions we have defined.

(i) $L_x(g) = \|g\| + 2d(x, A_g)$ if g is not an inversion.
(ii) $\|g\| = \max\{0, L_x(g^2) - L_x(g)\}$.

Here, it should be noted that for points $x \notin A_g$, there is a unique closest point of A_g to x; the distance between these points is the one referred to in (i). While $A_g = A_{g^n}$ for all $n \neq 0$ in the case where g is hyperbolic, if g fixes a point, it is possible that $A_{g^2} \supset A_g$. We may have $L_x(g^2) - L_x(g) < 0$ in this case.

Free actions are characterised, in the language of length functions, by the facts (i) $\|g\| > 0$ for all $g \neq 1$, and (ii) $L_x(g^2) > L_x(g)$ for all $g \neq 1$. The latter follows from the fact that $\|g^n\| = |n|\|g\|$ for all g.

We note that there are axioms for the translation length function which were shown to essentially characterise actions on Λ-trees, up to equivariant isometry, by Parry, [18].

Now consider a free product $G = *_{i\in I} G_i$ of groups G_i, and suppose that each G_i is Λ-free. Then, equivalently, each G_i admits a Lyndon length function L_i with the additional property that $L_i(g^2) > L_i(g)$ for all $g \in G_i$ and $i \in I$. We now define a length function L on G from the given length functions on the G_i. Given an arbitrary element $\gamma = g_1 g_2 \dots g_n$ of G (where each g_i and g_{i+1} are non-trivial and belong to distinct free factors) we put

$$L(\gamma) = \sum_{i=1}^{n} L_{j_i}(g_i).$$

It is easy to check that L satisfies L1, L2 and L3, and that $L(g^2) > L(g)$ for all $g \neq 1$. A somewhat tedious check establishes L4. By Chiswell's theorem, we can conclude that there is a free action of G on a Λ-tree.

This argument is not sufficient to show that a free product of tree-free groups is tree-free; in other words, if the ordered abelian group varies with $i \in I$, the definition of L above fails. However, using the base-change functor as described below, it is possible to produce a single ordered abelian group Λ from the Λ_i, and a set of suitable Λ-valued length functions so that the argument above works.

It is not initially clear that there are any other examples of Λ-free groups — Lyndon stated a conjecture in [15] to the effect that all $\mathbb{R}$-free groups can be accounted for in this way, and it was 11 years before a counterexample was found. It was shown by Morgan and Shalen in 1991 that if M is a closed surface other than the connected sum of at most 3 projective planes, then $\pi_1(M)$ admits a free action on an $\mathbb{R}$-tree.

An important device relating Λ_1-trees and Λ_2-trees is the *base change functor.*

Theorem 2.3 (Base-Change Functor)**.** *Let $h : \Lambda_1 \to \Lambda_2$ be an order preserving homomorphism between ordered abelian groups and let G be a group acting by isometries on a Λ_1-tree, (X_1, d_1). Then there is a Λ_2-tree, (X_2, d_2) on which G acts by isometries and a mapping $\phi : X_1 \to X_2$ such that*
(i) $d_2(\phi(x), \phi(y)) = h(d_1(x, y))$, for all $x, y \in X_1$,
(ii) $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X_1$,
(iii) $\|g\|_{X_2} = h(\|g\|_{X_1})$ for all $g \in G$.

The Λ_2-tree X_2 constructed in the proof is essentially the smallest possible, and is denoted $X_1 \otimes_{\Lambda_1} \Lambda_2$ even though the definition depends on the choice of h. It is an easy exercise to show that in general if the action of G on X_1 is free and h is injective, then the action of G on $X_2 = X_1 \otimes_{\Lambda_1} \Lambda_2$ is also free.

Let us consider some examples.

1: The barycentric subdivision of a Λ-tree

Let (X, d) be a Λ-tree, and h the endomorphism $\lambda \mapsto 2\lambda$. Then the base change functor gives rise to a Λ-tree X' with the property that $d'(\phi(x), \phi(y)) = 2d(x, y)$. It can be seen from the definition of segment that there exists a midpoint of $[\phi(x), \phi(y)]$; that is, a point u such that $d'(\phi(x), u) = d'(u, \phi(y)) = \lambda$.

The importance of this example arises from the fact that if σ is an inversion of X, and $[x, y]$ is stabilised by σ, then the induced isometry of X' must fix the midpoint. Thus, for any isometric action of G on a X, the associated action on X' is without inversions. Since h is an embedding, X embeds in X', and σ is hyperbolic as an isometry of X precisely when it is hyperbolic as an isometry of X'. Indeed ϕ is an equivariant embedding of X in X', so the actions on X and X' are essentially equivalent for most purposes. This is why it is often somewhat benignly assumed that an isometric action on a Λ-tree is without inversions.

2: Λ'-trees where $\Lambda' \leq \Lambda$

For such Λ', the base change functor applied to the inclusion map enables us to embed Λ'-trees equivariantly in Λ-trees, and enables us to view group actions Λ'-trees as a particular case of actions on Λ-trees.

In particular let (X, d) be a $\mathbb{Z}$-tree, and $h : \mathbb{Z} \longrightarrow \mathbb{R}$ the inclusion map. The base change functor gives rise to an $\mathbb{R}$-tree $\mathrm{real}(X)$, called the geometric realisation of X, which is usually defined to be a quotient of the union of closed intervals I_e where e ranges through the set of edges of the associated simplicial tree.

3: Λ'-trees where $h : \Lambda \twoheadrightarrow \Lambda'$

In each of the examples so far h has been an embedding. This need not always be so: in what follows we will consider the case where $h : \Lambda \longrightarrow \Lambda'$ is an (order-preserving) epimorphism. In this case, the kernel Λ_0 is a convex subgroup of Λ. (Conversely, for any convex subgroup Λ_0, the quotient group is itself an ordered abelian group onto which Λ maps via the natural map.)

It is easy to check that the relation $\sim$ on a Λ-tree given by

$$x \sim y \text{ if } d(x, y) \in \Lambda_0$$

is an equivalence relation. We call the equivalence classes *balls of radius* Λ_0. The convexity of Λ_0 means that such balls are subtrees. The distance between points belonging to distinct balls is larger than $n\lambda_0$ for any $\lambda_0 \in \Lambda_0$ and $n \in \mathbb{Z}$; we think of this distance as being 'infinitely larger' than any element of Λ_0.

The Λ'-tree arising from h may now be thought of as obtained from the Λ-tree by collapsing each of the balls of radius Λ_0 to a point. In the opposite direction the Λ-tree X is, in a certain sense, built up from the Λ'-tree X' and the balls of radius Λ_0, which are themselves naturally Λ_0-trees.

3. Group actions on $(\mathbb{Z} \times \Lambda_0)$-trees

We describe in this section some results for groups acting isometrically on a Λ-tree, where Λ is of the form $\mathbb{Z} \times \Lambda_0$ for some ordered abelian group Λ_0 and where the product is given the lexicographic ordering. One has a map from $\mathbb{Z} \times \Lambda_0$ to $\mathbb{Z}$, as in example 3, by factoring out Λ_0 and we can apply the base change functor, via this map to deduce the following.

Theorem 3.1. *Let G be a group acting isometrically on a $\mathbb{Z}\times\Lambda_0$-tree. Then G acts on a $\mathbb{Z}$-tree (a simplicial tree). Hence, by Bass-Serre theory, G has a graph of groups decomposition where each vertex group acts isometrically on some Λ_0 tree and there is a homomorphism from each edge group into Λ_0. If, additionally, the action is free, then each vertex group is Λ_0-free and each each edge group is a maximal abelian subgroup of G that embeds into Λ_0.*

Let us give some justification for this result. The effect of applying the base change functor, using the natural map $\mathbb{Z} \times \Lambda_0 \to \mathbb{Z}$, is to form a $\mathbb{Z}$-tree X^* by collapsing all the balls of radius Λ_0. Each of these balls is a Λ_0-tree and these form the vertex set of X^*. Hence the vertex stabilisers of X^* act on Λ_0-trees.

An edge of X^* corresponds to a pair of *ends of full Λ_0 type* in adjacent Λ_0 balls. Here an end of full Λ_0-type is defined by first defining a half Λ_0-line to be the isometric image of $[0, \infty) \subseteq \Lambda_0$ and two such half lines are said to be equivalent if they meet in another half line. The ends of full Λ_0-type are then the equivalence classes. The definition of general ends is slightly more technical, but the important feature is that, in our situation, a vertex group G_v has an associated Λ_0-tree, X_v, and acts on the set of ends $Ends(X_v)$.

Also, given any end $\epsilon \in Ends(X_v)$, there is an induced homomorphism, τ_ϵ, from the stabiliser of ϵ to Λ_0, $\tau_\epsilon : G_\epsilon \to \Lambda_0$. In fact, $\tau_\epsilon(g) = \pm\|g\|$, where the sign depends on whether g translates toward ϵ or away from it.

Thus, in the $\mathbb{Z}$-tree, X^*, edge groups are end stabilisers. In the case of a free action, the vertex stabilisers act freely on Λ_0 trees and the maps τ_ϵ become monomorphisms.

It then becomes natural to ask whether there is a converse to this result. That is, given a group, G, in terms of a graph of groups decomposition, associated Λ_0-trees and various other conditions, when can we contruct an action of G on a $(\mathbb{Z} \times \Lambda_0)$-tree? This is answered by results of Bass.

We introduce first some notation and conventions. We regard a tree X^* as a set of vertices to which are associated its set of directed edges $e = (x, y)$. We write $\partial_0(e) = x$ and $\partial_1(e) = \partial_0(\bar{e}) = y$ for such an edge.

Part of the structure of a graph of groups $(\mathcal{G}, Y^*)$ is an embedding of each edge group $\mathcal{G}(e)$ in its end vertex group $\mathcal{G}(\partial_0(e))$ — we denote such an embedding by α_e.

Theorem 3.2 (Bass 3.8)**.** *Let $(\mathcal{G}, Y^*)$ be a graph of groups with fundamental group G, and for vertices or edges s of Y^* let G_s denote the corresponding group. For (vertices) $x^* \in Y^*$, let X_{x^*} be a Λ_0-tree on which G_{x^*} acts.*

For each edge e of Y^*, let ϵ_e be an end of X_{x^*} of full Λ_0-type, and τ_e the homomorphism $G_{\epsilon_e} \longrightarrow \Lambda_0$.

Suppose that the following are also satisfied.

(i) *For edges e of Y^*, the end stabiliser $(G_{\partial_0(e)})_{\epsilon_e}$ is equal to $\alpha_e(G_e)$ and*
$$\tau_e \circ \alpha_e = -\tau_{\bar{e}} \circ \alpha_{\bar{e}}.$$
(ii) *For distinct edges e and f of Y^* with $\partial_0(e) = \partial_0(f)$, the ends ϵ_e and ϵ_f lie in distinct $G_{\partial_0(e)}$ orbits.*

There is a Λ-tree X on which G acts, with $X^* \cong X \otimes_\Lambda \mathbb{Z}$.

Remarks: (a) Assumption (i) guarantees that if g fixes an edge $e = (x^*, y^*)$of X^* the translation length and orientation of g, as an element of both G_{x^*} and G_{y^*}, are compatible. With regard to the requirement that each ϵ_e is of full Λ_0-type, it is worth noting that to any Λ'-tree X', there is the associated Λ'-fulfilment $\lambda(X')$, which is a Λ'-tree in which X' embeds $\mathrm{Aut}(X')$-equivariantly and of which every end is of full Λ'-type. See [2, E1] for details.

(b) Note that the metric on X is essentially determined by the data required by the hypotheses. There is however some degree of freedom in terms of what Bass calls *end maps*.

(c) It is easy to see that the resulting action of G on X is free if and only if the action of each vertex group G_{x^*} on $X(x^*)$ is free.

Armed with this result, we can give many examples of free actions on Λ-trees. In the following corollary, the group G is the fundamental group of a graph of groups with a single vertex and a single edge loop. Since maximal abelian subgroups correspond to line stabilisers, one can associate the two ends of this line to the two oriented edges to apply the previous result and get the following. Bass calls the group G a 'benign' HNN extension.

Corollary 3.3 ([2], [11], [12])**.** *Let G_0 be a group that admits a free action on a Λ_0-tree, H a maximal abelian subgroup, and put*
$$G = \langle G_0, t \mid t^{-1}ht = h \textit{ for all } h \in H\rangle.$$
Then G acts freely on a $(\mathbb{Z} \times \Lambda_0)$-tree.

4. $\mathbb{Z}^n$-FREE GROUPS

Since $\mathbb{Z}$-free groups are precisely free groups, the results of the previous section are particularly powerful when Λ_0 is a cartesian product of a finite number of copies of $\mathbb{Z}$. That is, when we are considering $\mathbb{Z}^n$-actions. Of particular interest is the case of free $\mathbb{Z}^n$-actions and $\mathbb{Z}^n$-free groups.

One motivation for considering this class of groups is that one has some good examples. For instance, using a construction involving foliations on surfaces and taking the fundamental group of a finite collection of surfaces, glued together at the boundary, Liousse was able to prove the following.

Theorem 4.1 ([13]). *A group is said to act affinely on an $\mathbb{R}$-tree if to each group element, g, there is associated a postive constant, μ_g, such that $d(gx, gy) = \mu_g d(x, y)$. Then there are groups which act freely and affinely on $\mathbb{R}$-trees, but admit no free isometric action on an $\mathbb{R}$-tree.*

However, it turns out [16] that all these groups acting freely and isometrically on $(\mathbb{Z} \times \mathbb{Z})$-trees.

One of the most important examples of groups in this class have been considered by Kharlampovich and Myasnikov, who have proved the following.

Recall that a group G is said to be *fully residually free* if, for any finite set of non-trivial elements $g_1, \ldots, g_k$, there exists a homomorphism, $\phi : G \to F$, to a free group F, such that $\phi(g_i) \neq 1$ for all i.

Theorem 4.2 ([11], [12]). *Let G be a finitely generated fully residually free group. Then G acts freely on some $\mathbb{Z}^n$-tree.*

Indeed, they proved a conjecture that every finitely generated fully residually free group embeds in Lyndon's exponential group $F^{\mathbb{Z}[x]}$, which is the union of the groups G_n, where $G_0 = F$ is a free group and

$$G_{n+1} = \langle G_n, t_n \mid t_n^{-1} u_n t_n = u_n,\ u_n \in C_n \rangle$$

where C_n is some proper centraliser in G_n. Since G_0 acts freely on a $\mathbb{Z}$-tree, inductively G_n acts freely on a $(\mathbb{Z}^{n+1})$-tree — this hinges on the observation that the maximal abelian subgroups of G_n are exactly the proper centralisers in G_n. Since every finitely generated subgroup of $F^{\mathbb{Z}[x]}$ is a subgroup of some G_n, this proves the theorem.

We note that Sela in [20] proves that any finitely generated fully residually free group is $\mathbb{R}^n$-free for some n. However, to answer Conjecture Q 3.4 on Bestvina's problem page in the negative, it is not true that any $\mathbb{R}^n$-free (or even $\mathbb{Z}^n$-free) group is fully residually free.

Theorem 4.3 ([8], [14])**.** *The fundamental group G of the connected sum of three projective planes is $(\mathbb{Z} \times \mathbb{Z})$-free. However, any homomorphism from G to a free group has a cyclic image. Hence G is not fully residually free.*

We note that while Theorem 4.2 proves that most surface groups are $\mathbb{Z}^n$-free, [16] prove that every surface group, except the fundamental group of the projective plane and the the Klein bottle is $(\mathbb{Z} \times \mathbb{Z})$-free. And it is easy to see that the stated two exceptions cannot act freely on any Λ-tree. Moreover, results from [16] indicate that the class of $\mathbb{Z}^n$-free groups is much larger than the class of fully residually free groups.

Theorem 4.4. *Let G_1 and G_2 be $\mathbb{Z}^n$-free groups. Then the amalgamated free product $G_1 *_{\mathbb{Z}} G_2$ is $\mathbb{Z}^m$-free for some m, provided that the image of $\mathbb{Z}$ is maximal abelian in each G_i.*

One would expect that the proof of the above theorem would consist of 'rescaling' the element generating the infinite cyclic subgroup in each factor. However, this naive approach is not quite sufficient in the non-Archimedean case. For instance, let us suppose that both G_1 and G_2 are $\mathbb{Z}^3$-free and denote by g_i the generator in G_i of the image of $\mathbb{Z}$ $(i = 1, 2)$. Then we may have that for the given actions, $\|g_1\| = (1, 0, 0)$, while $\|g_2\| = (0, 1, -1)$, both of which are positive elements of $\mathbb{Z}^3$. In order to apply Theorem 3.2, it is essentially sufficient to find actions in which g_1 and g_2 have the same $\mathbb{Z}^n$ translation length. The first problem is that they are not even comparable, but this is easily managed by embedding $\mathbb{Z}^3$ into $\mathbb{Z}^4$ in two different ways and applying the base change functor. One would then get that $\|g_1\| = (0, 1, 0, 0)$ and $\|g_2\| = (0, 1, -1, 0)$. But one is left with the problem that rescaling, even applied to coordinates, can only be multiplication by a positive element and the result has to be obtained by defining a new translation length function or a Lyndon length function from the old.

Although we are not aware that the following fact has been recorded, it is fairly straightforward to show that the class of groups which are finitely generated and act freely on some $\mathbb{Z}^n$-tree have solvable word problems. It follows from the following elementary fact about graphs of groups.

Theorem 4.5. *Let $\mathcal{G}$ be a finite graph of groups in which all the vertex groups have solvable word problem and in which each edge group has solvable membership problem in its respective vertex groups. Then $\pi_1(\mathcal{G})$ has solvable word problem.*

One can use this to show that any finitely generated $\mathbb{Z}^n$-free group has solvable word problem by induction on n. The case $n = 1$ corresponds to the class of finitely generated free groups where the result is well known. Suppose the then that we have shown that the class of finitely generated

$\mathbb{Z}^{n-1}$-free groups have solvable word problem and consider a finitely generated $\mathbb{Z}^n$-free group G. By Theorem 3.1, $G = \pi_1(\mathcal{G})$ and the vertex groups must be $\mathbb{Z}^{n-1}$-free groups. Since G is finitely generated, we can take $\mathcal{G}$ to be finite. Moreover, the vertex groups must be finitely generated and hence, by induction, have solvable word problem. The edge groups are maximal abelian subgroups in their vertex groups and must, by the CSA property Theorem 2.1, have solvable membership problem. (If M is maximal abelian in a tree-free group, then the CSA property says that $g \in M$ if and only if g commutes with some fixed non-trivial element of M. This is decidable from the word problem.) Thus the hypotheses of the above Theorem would hold and G would has solvable word problem.

Note that while $\mathbb{Z}^n$-free groups have all the properties of general tree-free groups, some of which we have listed in Theorem 2.1, one can also show that they have more.

(i) The class of $\mathbb{Z}^n$-free groups is closed under taking free products.
(ii) Any abelian subgroup of a $\mathbb{Z}^n$-free group is free abelian of rank at most n.
(iii) $\mathbb{Z}^n$-free groups are coherent. That is, any finitely generated subgroup of a $\mathbb{Z}^n$-free group is finitely presented.
(iv) A finitely generated $\mathbb{Z}^n$-free group, G, is of type $\mathcal{F}$. This means that G is the fundamental group of a finite cell complex whose universal cover is contractible and the same is true of any finitely generated subgroup.
(v) A finitely generated $\mathbb{Z}^n$-group with no $\mathbb{Z} \times \mathbb{Z}$ subgroups is word hyperbolic, as are all its finitely generated subgroups.
(vi) The class of $\mathbb{Z}^n$-free groups have solvable word problem.

Note that (i) and (ii) are restatements of (ix) and (xii) of Theorem 2.1. Items (iii) and (iv) are consequences of Theorem 3.1, as is (v) once one applies the Combination Theorem of [3]. Item (vi) is proved above.

In light of these results it seems fairly natural to ask if $\mathbb{Z}^n$-groups act properly and cocompactly on CAT(0) spaces (see [5]), as they share many of the same properties. While that question remains open, it is clear that not every group which acts properly and cocompactly on a CAT(0) space is $\mathbb{Z}^n$-free. For instance, the direct product of two non-abelian free groups of finite rank cannot be tree-free, because that group is not commutative transitive, but does act on a CAT(0) space, properly and cocompactly.

The theory of $\mathbb{Z}^n$-free groups seems much more tractible than the general theory of tree-free groups, and so one is led to ask the following.

Question 4.6. *Is every finitely generated tree-free group $\mathbb{Z}^n$-free for some n?*

A positive answer to this question would clearly simplify the study of tree-free groups and a useful first step in trying to prove it may be to show that $\mathbb{R}^m$-free groups are $\mathbb{Z}^n$-free for some n, perhaps by using the Rips machine.

An affirmative answer to this question (together with property (iii) of $\mathbb{Z}^n$-free groups) would, in particular, imply an affirmative answer to the following.

Question 4.7. *Is every finitely generated tree-free group finitely presented?*

As we have noted, all of the groups described in [13] which admit a free affine action on an $\mathbb{R}$-tree admit a free isometric action on a $\mathbb{Z} \times \mathbb{Z}$-tree. It is natural to wonder whether this reflects a general fact.

Question 4.8. *Suppose that G admits a free affine action on an $\mathbb{R}$-tree. Is G $\mathbb{Z}^n$-free?*

Since many of the groups we have discussed admit free actions, not merely on $\mathbb{Z}^n$-trees for some n, but on $(\mathbb{Z}\times\mathbb{Z})$-trees, it seems worthwhile to produce examples of groups that admit 'essential' free actions on $\mathbb{Z}^n$-trees.

Problem 4.9. *Describe a family of groups G_n ($n \in \mathbb{N}$) such that for all $n \in \mathbb{N}$*

(i) *G_n admits a free action on a $\mathbb{Z}^n$-tree;*
(ii) *G_n admits no free action on a $\mathbb{Z}^m$-tree for $m < n$;*
(iii) *G_n contains no free abelian subgroup of rank 2.*

Observe that such groups G_n would be word hyperbolic, by the Combination Theorem of [3].

We note that Theorem 4.4 has no direct analogue for HNN extensions. Indeed, it can be shown that a group of the form,

$$\langle x, y, t \mid t^{-1}[x^m, y^n]t = [x^s, y^t]\rangle$$

is tree-free if and only if it is $(\mathbb{Z}\times\mathbb{Z})$-free, which is if and only if $(|m|-|s|)$ and $(|t|-|n|)$ have the same sign. This leads us to ask for what HNN extensions of free groups with cyclic edge group admit a free action on a Λ-tree.

Question 4.10. *Are the following equivalent?*

(i) *A free group F acts freely on some $\mathbb{Z}$-tree, in which the group elements $u, v \in F$ have the same translation length.*
(ii) *The HNN extension $\langle F, t \mid t^{-1}ut = v\rangle$ is tree-free.*

Note that (i) implies that the HNN extension given in (ii) is actually $(\mathbb{Z} \times \mathbb{Z})$-free, but it seems hard to find an essential counterexample to this question.

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