Some free actions on non-archimedean trees

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Abstract

In this paper we show that various groups are $\mathbb{Z}^n$-free. In particular we show that almost every surface group is $(\mathbb{Z} \times \mathbb{Z})$-free as are the groups of Liousse [11]. We also demonstrate that the class of $\mathbb{Z}^n$-free groups is closed under taking amalgamated free products over an infinite cyclic group as long as it is maximal abelian in each vertex group. It follows that a large class of hyperbolic groups is $\mathbb{Z}^n$-free.

1 Introduction

Bass-Serre theory ([15], [6], [5]), the theory of group actions on simplicial trees, has proven to be very successful in describing the structure of discrete groups. In particular one can obtain a presentation for a group acting on a tree in terms of the vertex and edge stabilisers, a *graph of groups decomposition*.

It is therefore natural to try to generalise the notion of tree in order to broaden the scope of the theory. The study of isometric actions on real trees has had some success, notably Rips' Theorem which characterises finitely generated groups that admit a free isometric action on an real tree ([7], [8], [4]), and has become an established technique in geometric group theory.

There has been a further generalisation to group actions on Λ-trees, where Λ is an ordered abelian group. This generalisation encompasses actions on simplicial and real trees on taking $\Lambda = \mathbb{Z}$ and $\Lambda = \mathbb{R}$ respectively.

While several authors have extended many of the results from the theory of real trees to that of Λ-trees ([4] provides a comprehensive exposition) this has not been as widely adopted as a technique in the study of discrete groups as has the theory of real trees. This is due in part to an incomplete understanding of free actions on Λ-trees, and in part to a perceived lack of motivation in passing from $\mathbb{R}$ to a more general Λ.

The goal of this paper is to provide examples of groups which act freely on $\mathbb{Z}^n$-trees for some n. The original motivation was a paper of Liousse which, among other things, constructs some groups (described in 2.3) that admit free affine actions on real trees, but which do not admit any free isometric action on a real tree.

We describe in Theorem 3.1 a family of one-relator groups that act freely and isometrically on $(\mathbb{Z} \times \mathbb{Z})$-trees. As a consequence, we have

Corollary 3.4 *Liousse's groups are* $(\mathbb{Z} \times \mathbb{Z})$*-free.*

Thus by considering more general Λ we can construct actions of Liousse's groups which are, in some sense, better behaved.

Another consequence of Theorem 3.1 is

Corollary 3.3 *Every surface group except* $\pi_1(\mathbb{RP})$ *and* $\pi_1(\mathbb{RP}\#\mathbb{RP})$ *is* $(\mathbb{Z} \times \mathbb{Z})$*-free. Hence every finitely generated* $\mathbb{R}$*-free group is* $\mathbb{Z}^n$*-free for some* n*. Moreover we can take* $n = \max\{2, ab(G)\}$*, where* $ab(G)$ *is the maximal rank of any free abelian subgroup of* G*.*

We note that Gaglione and Spellman [9] have shown that $\pi_1(M)$ is $(\mathbb{Z} \times \mathbb{Z})$-free when $M = \mathbb{RP}\#\mathbb{RP}\#\mathbb{RP}$. Also, it has been shown in [10], that finitely generated fully residually free groups are $\mathbb{Z}^n$-free, for some n. As every surface group except the fundamental group of the connected sum of at most three projective planes is fully residually free, Corollary 3.3 is largely known, though we believe the bound on n to be new.

The main result of the paper, a consequence of Theorem 4.3, is

Theorem *Suppose that G_1 and G_2 are $\mathbb{Z}^n$-free for some n. Then $G_1 *_{\mathbb{Z}} G_2$ is $\mathbb{Z}^m$-free for some $m \geq n$ if and only if $\mathbb{Z}$ embeds as a maximal abelian subgroup in each G_i.*

A result of Bestvina and Feighn, [3], implies that if, in addition, G_1 and G_2 are word hyperbolic, then $G_1 *_{\mathbb{Z}} G_2$ is also word hyperbolic if $\mathbb{Z}$ is maximal abelian in each G_i. Thus one may construct many word hyperbolic groups which act freely on some $\mathbb{Z}^n$-tree.

Throughout, we rely on results from Bass's paper [2] which we recall in 2.4. The main difficulty in applying these results is the presence of technical hypotheses concerning the particular actions on $\mathbb{Z}^n$-trees.

Our main theorem does not have an exact analogue in the case of HNN-extensions. In fact, the group

$$\langle x, y, t \mid t^{-1}[x, y]t = [x^2, y^2]\rangle$$

is not Λ-free for any Λ.

2 Preliminaries

2.1 Λ-trees

We refer the reader to [4] for a full account of the basic theory of Λ-trees and isometric actions thereon. We recall some of the details here.

Let Λ be a (linearly) ordered abelian group . Examples of ordered abelian groups include $\mathbb{R}$, and $\mathbb{Z}^n$ equipped with the lexicographic order.

A Λ-metric space is defined in the same way as a conventional metric space with $\mathbb{R}$ replaced by Λ. Thus Λ is itself a Λ-metric space by putting $d(\lambda, \mu) = |\lambda - \mu| = \max\{\lambda - \mu, \mu - \lambda\}$. A Λ-metric space X is *geodesically convex* if, for all $x, y \in X$, there is a unique isometry $\phi : [0, d(x, y)]_\Lambda \longrightarrow X$ with $\phi(0) = x$ and $\phi(d(x, y)) = y$. We denote the image of such an isometry by $[x, y] = [x, y]_X$.

The Λ-metric space (X, d) is a Λ-*tree* if

(i). X is geodesically convex;

(ii). $[x, y] \cap [y, z] = [y, w]$ for some $w \in X$;

(iii). $[x, y] \cap [y, z] = \{y\}$ implies $[x, y] \cup [y, z] = [x, z]$.

A Λ-tree can also be characterised as a geodesically convex Λ-metric space which is 0-hyperbolic and which satisfies $d(x,v)+d(y,v)-d(x,y)\in 2\Lambda$ for all $x,y,v\in X$.

Associated to an isometric group action on a Λ-tree is the *translation length function* $\|.\|: G\to\Lambda$. The value of $\|g\|$ is equal to 0 if g has a fixed point or is an inversion, and is equal to $\min\{d(x,gx): x\in X\}$ otherwise. We call an action of G on a Λ-tree *free* if there are no inversions and only the identity element has a fixed point. An action is thus free if and only if $\|g\|>0$ for all $g\in G\backslash\{1\}$.

A group that admits a free action on a $\mathbb{Z}$-tree is a free group; this fact follows, for example, from Bass-Serre theory. A group G is said to be Λ-*free* if G admits a free isometric action (without inversions) on some Λ-tree, and *tree-free* if it is Λ-free for some Λ. It is an elementary fact that tree-free groups are torsion free and *commutative transitive*, that is, commutativity is a transitive relation on non-identity elements. Clearly, any subgroup of a Λ-free group is Λ-free and it can also be shown that free products of Λ-free groups are Λ-free.

Note that Λ, viewed as a Λ-metric space, is a Λ-tree. Moreover, Λ acts on itself by translations and hence is Λ-free.

The natural and long-standing question of which groups are $\mathbb{R}$-free was answered, in the finitely generated case, by Rips in 1991:

Theorem 2.1.1 (Rips' Theorem) *A finitely generated group G admits a free action on an $\mathbb{R}$-tree if and only if G is expressible as a free product $G_1 * G_2 * \cdots G_n$ where each G_i is either a finitely generated free abelian group or the fundamental group of a closed surface other than the connected sum of* 1, 2 *or* 3 *projective planes.*

2.2 Surface groups

There is a well-known classification of closed surfaces (i.e. compact, connected 2-manifolds without boundary) — see [13] for details. It follows from this classification that the fundamental group of an orientable surface M of genus g has the presentation

$$\pi_1(M)=\langle x_1,y_1,\ldots,x_g,y_g|\prod_{i=1}^{g}[x_i,y_i]=1\rangle$$

and the fundamental group of a non-orientable surface of genus g has the presentation

$$\pi_1(M)=\langle x_1,\ldots,x_g|\prod_{i=1}^{g}x_i^2=1\rangle.$$

2.3 Affine actions and Liousse groups

Let X be an $\mathbb{R}$-tree. An action of G on X is said to be *affine* if, for each $g\in G$, there is a positive real number α_g such that $d(gx,gy)=\alpha_g d(x,y)$ for all $x,y\in X$. In her paper [11], Liousse examines affine actions on $\mathbb{R}$-trees and constructs groups which admit a free, affine action on an $\mathbb{R}$-tree, but do not act freely and isometrically on an $\mathbb{R}$-tree. The groups considered fall into two Types.

Type I groups are obtained by taking a finite collection of surfaces, taking one boundary component from each, and gluing the surfaces along these boundary components via homeomorphisms. The fundamental group of the resulting quotient space is a Type I group.

It follows from Van Kampen's theorem (and using Tietze transformations, if necessary) that Type I groups are isomorphic to an iterated free product with amalgamation of the form, $G = F_1 *_{\mathbb{Z}} F_2 *_{\mathbb{Z}} * \cdots *_{\mathbb{Z}} F_n$, where each F_i is a finitely generated free group. In fact, each amalgamated $\mathbb{Z}$ will be generated by the same element of G since the embeddings in F_i have the same image for $2 \leq i \leq n-1$, though this will not be relevant for our purposes. The embedded image of each cyclic group in the appropriate free group is a product of commutators, or a product of (at least two) squares, of basis elements. In particular, these images are maximal abelian in the respective free groups.

Type II groups are formed by taking two surfaces and gluing a boundary component of one to a boundary component of the other via a map of some degree $k \geq 1$ (both of these boundary components are circles). The fundamental group of the resulting space is a Type II group and is either a free group or has a presentation of the form $\langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid w = v^k \rangle$. Here w corresponds to the path around the boundary component of the first surface and is of the form $[x_1, x_2] \ldots [x_{2r-1}, x_{2r}]$ where $n = 2r$, or $x_1^2 \ldots x_n^2$ and $n \geq 2$. Similarly v corresponds to the path around the boundary component of the second surface. For our purposes it suffices to observe that v is a non-trivial word in $y_1, \ldots, y_m$.

2.4 Free actions on $(\mathbb{Z} \times \Lambda)$-trees

We shall be concerned with showing that certain groups are $\mathbb{Z}^n$-free. In order to do this we shall use results from [2] which we collect here for convenience.

Theorem 2.4.1 (Bass, [2]) *(i). Suppose that G acts freely on a Λ-tree X and that $t_1, t_2 \in G$ each generates a maximal abelian subgroup of G. Further assume that t_1 is not conjugate to t_2^{-1} and that $\|t_1\|_X = \|t_2\|_X$. Then the HNN-extension $\langle G, x \mid x^{-1} t_1 x = t_2 \rangle$ is $(\mathbb{Z} \times \Lambda)$-free.*

(ii). Suppose that G_1, G_2 act freely on Λ-trees X_1, X_2 respectively and that $t_1 \in G_1, t_2 \in G_2$ generate maximal abelian subgroups. Further, if we assume that $\|t_1\|_{X_1} = \|t_2\|_{X_2}$, then the amalgamated free product $G_1 *_{\langle t_1 = t_2 \rangle} G_2$ is $(\mathbb{Z} \times \Lambda)$-free.

(iii). Suppose that $G_1, \ldots, G_k$ are free groups, and let $s_i \in G_i$ $(1 \leq i \leq k-1)$ and $t_i \in G_i$ $(2 \leq i \leq k)$ be two families of elements, none of which is a proper power in their respective group. Then the amalgamated free product $G_1 *_{\langle s_1 = t_2 \rangle} G_2 *_{\langle s_2 = t_3 \rangle} * \cdots *_{\langle s_{k-1} = t_k \rangle} G_k$ is $(\mathbb{Z} \times \mathbb{Z})$-free.

Part (i) of the theorem is Proposition 4.15 of [2]. Part (ii) follows from [2], Theorem 4.7 in the same way that Proposition 4.15 follows from Theorem 4.7.

Namely, one replaces each X_i by its Λ-fulfilment (see the appendix on ends in [2]) and applies Theorem 4.7 of [2].

Let ϵ_1 denote the attracting end of $A_{t_1} \subseteq X_1$, and ϵ_2 the repelling end of $A_{t_2} \subseteq X_2$. Then the stabiliser in G_i of ϵ_i is equal to $\langle t_i \rangle$ by Theorem 1.10 of [2] since this subgroup is maximal abelian in G_i, so that condition α of Bass's Theorem 4.7 is satisfied.

The condition that $\|t_1\|_{X_1} = \|t_2\|_{X_2}$ and choice of ends ϵ_i ensures that we have $\tau_{\epsilon_1} = -\tau_{\epsilon_2}$, and so Bass's condition β is satisfied.

Finally, Bass's condition γ is trivially satisfied, since there is only one unoriented edge in our graph of groups.

Part (iii) of the theorem follows from Theorem 4.9 of [2] once we note that free groups are precisely the $\mathbb{Z}$-free groups, that the conditions on s_i, t_i ensure that the subgroups they generate are maximal abelian, and that condition (iii)(γ) of Theorem 4.9 is trivially satisfied.

3 Some $(\mathbb{Z} \times \mathbb{Z})$-free groups

In this section we shall show that Liousse groups are $(\mathbb{Z} \times \mathbb{Z})$-free. We also show that surface groups are $(\mathbb{Z} \times \mathbb{Z})$-free and hence that finitely generated $\mathbb{R}$-free groups are $\mathbb{Z}^n$-free. These will be easy consequences of the following result.

Theorem 3.1 *Let $G = \langle a, b, x_1, \ldots, x_n \mid aba^{-1}b^{\epsilon} = w \rangle$ where $\epsilon = \pm 1$ and w is any word in $\{x_1, \ldots, x_n\}$. Then G acts freely on a $(\mathbb{Z} \times \mathbb{Z})$-tree, unless $\epsilon = 1$ and $w = 1$.*

Proof: Note first that if $w = 1$ and $\epsilon = -1$, then G has the sole defining relation $[a, b] = 1$ and is thus $(\mathbb{Z} \times \mathbb{Z})$-free since it is a free product of a free abelian group of rank 2 with a free group. We will henceforth assume, therefore, that $w \neq 1$.

By the Freiheitssatz [12], the set $\{b, x_1, \ldots, x_n\}$ is a basis for a free subgroup of G which we will denote by F.

We first consider the case where w has even length in the given generators of F. Thus we can write $w = w_1 w_2$ where this product is reduced as written and w_1, w_2 have the same length. Now consider a new basis $X = \{c, x_1 \ldots, x_n\}$ of F given by $c = w_2 b^{-\epsilon}$. As w does not involve b, the lengths of w_1, w_2 are equal with respect to this new basis. Replacing b by $({w_2}^{-1}c)^{-\epsilon}$ we get a new presentation of G,

$$G = \langle a, c, x_1, \ldots, x_n \mid a({w_2}^{-1}c)^{-\epsilon}a^{-1} = w_1 c \rangle.$$

This realises G as an HNN-extension of F by the stable letter a and with cyclic edge group.

Clearly, $({w_2}^{-1}c)^{-\epsilon}$ and $w_1 c$ have the same word length with respect to X and they are both cyclically reduced since they each contain exactly one occurrence of the basis letter c either at the start or at the end. Now let Γ be the Cayley graph of F with respect to the basis X. We have shown that $(w_2{}^{-1}c)^{-\epsilon}$ and w_1c have the same translation length when considered as isometries of Γ.

Recall that cyclically reduced conjugates in a free group are cyclic permutations of each other. Thus if $(w_2{}^{-1}c)^{\epsilon}$ and w_1c were conjugate in F, it would force $\epsilon = 1$ and $w_2{}^{-1} = w_1$. As the latter is not the case, we deduce that $(w_2{}^{-1}c)^{\epsilon}$ and w_1c are not conjugate in F. Thus we may apply part (i) of Theorem 2.4.1 to conclude that G is $(\mathbb{Z} \times \mathbb{Z})$-free.

In the case where w is not of even length, we consider the group

$$G_1 = \langle a, b, y_1, \ldots, y_n \mid aba^{-1}b^{\epsilon} = v \rangle$$

where the word v is obtained from w by replacing each occurrence of x_i by y_i^2. We then map G to G_1 by sending a to a, b to b and x_i to y_i^2; it is easy to see that this is an embedding. We can apply the above argument to show that G_1 acts freely on a $(\mathbb{Z} \times \mathbb{Z})$-tree, and G acts by restriction.

Corollary 3.2 *Let G be a group with presentation $G = \langle X_1 \sqcup X_2 \mid w_1 = w_2^n \rangle$ where $n \geq 1$, and where each w_i is either a product $x_1^2 x_2^2 \ldots x_{m_i}^2$ of squares of distinct elements of X_i, or a product $[x_1, y_1][x_2, y_2] \ldots [x_{m_i}, y_{m_i}]$ of commutators of distinct elements of X_i. Assume that w_1 is non-trivial and not equal to x_1^2, and that $w_2 \neq 1$ if $w_1 = x_1^2 x_2^2$. Then G is $(\mathbb{Z} \times \mathbb{Z})$-free.*

Proof: If w_1 is a product of commutators, then the relation can be rewritten as $[x_1, y_1] = w_2^n [y_{m_1}, x_{m_1}] \ldots [y_2, x_2]$. The right-hand side of this equation has no occurrence of x_1 or y_1, and so, by Theorem 3.1, we conclude that G is $(\mathbb{Z} \times \mathbb{Z})$-free.

If $w_1 = x_1^2 x_2^2 \ldots x_{m_1}^2$ where $m_1 \geq 2$, then putting $a = x_1 x_2$, G can be written in the form

$$\langle X_1' \sqcup X_2 \mid x_1 a x_1^{-1} a = w_2^n x_n^{-2} \ldots x_3^{-2} \rangle,$$

where $X_1' = X_1 \cup \{a\} \backslash \{x_2\}$. Once again, we can apply Theorem 3.1 (noting that $w_2^n x_{m_1}^{-2} \cdots x_3^{-2} \neq 1$ by assumption) to conclude that G is $(\mathbb{Z} \times \mathbb{Z})$-free.

It follows from the corollary that $\pi_1(M)$ is $(\mathbb{Z} \times \mathbb{Z})$-free for any closed surface M except for $\mathbb{RP}$ and $\mathbb{RP}\#\mathbb{RP}$.

Moreover these exceptions are genuine: $\pi_1(\mathbb{RP})$ is not torsion-free, and in the group $\pi_1(\mathbb{RP}\#\mathbb{RP})$ we have $x^2y^2 = 1 = y^2x^2$ which, in a tree-free group, would force $xy = yx$ by commutative transitivity. It follows that neither of these groups is tree-free. The latter example also shows that Theorem 3.1 is false if the case $\epsilon = 1$, $w = 1$ is not excluded.

It is clear that finitely generated free abelian groups of rank n are $\mathbb{Z}^n$-free and that a $\mathbb{Z}^m$-free group is also $\mathbb{Z}^n$-free for any $n \geq m$ — the latter can also be seen as a consequence of Theorem 4.1. Moreover the class of Λ-free groups is closed under the formation of free products. One can thus deduce the following from Rips' theorem.

Corollary 3.3 *Every surface group except $\pi_1(\mathbb{RP})$ and $\pi_1(\mathbb{RP}\#\mathbb{RP})$ is $(\mathbb{Z}\times\mathbb{Z})$-free. Hence every finitely generated $\mathbb{R}$-free group is $\mathbb{Z}^n$-free for some n. Moreover we can take $n = \max\{2, ab(G)\}$, where $ab(G)$ is the maximal rank of any free abelian subgroup of G.*

Corollary 3.4 *Liousse groups are $(\mathbb{Z}\times\mathbb{Z})$-free.*

Proof: A group of Type I is $(\mathbb{Z}\times\mathbb{Z})$-free by part (iii) of Theorem 2.4.1.
An examination of the presentation of the Type II groups shows that they are $(\mathbb{Z}\times\mathbb{Z})$-free by Corollary 3.2.

4 Constructing $\mathbb{Z}^n$-free groups

In this section we show that certain amalgamated free products $G_1 *_{\mathbb{Z}} G_2$ are $\mathbb{Z}^n$-free. In turn, this allows us to construct a large class of hyperbolic groups which are $\mathbb{Z}^n$-free.

In order to do this, we apply part (ii) of Theorem 2.4.1. However, this theorem relies on certain translation lengths being equal. Thus, starting with groups G_1 acting freely on a $\mathbb{Z}^{n_1}$-tree and G_2 acting freely on a $\mathbb{Z}^{n_2}$-tree, we show how one can modify the actions in order to make them sufficiently compatible to apply the result and deduce that $G_1 *_{\mathbb{Z}} G_2$ is $\mathbb{Z}^n$-free for some n.

Before proceeding, we introduce some notation. We write e_j to denote the element $(\underbrace{0,\dots 0}_{j-1},1,0,\dots 0)$ of $\mathbb{Z}^n$ for $1 \le j \le n$. We write $\mathbb{Z}^k$ to denote not only the ordered abelian group $\underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{k}$ but also the subgroup $0\times\cdots 0\times \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{k}$ of $\mathbb{Z}^n$ for $k \le n$.

In what follows it will be convenient to be able to change our actions by scaling the metric and thinking of $\mathbb{Z}^n$ as a subgroup of $\mathbb{Z}^{n+k}$. These modifications to actions on Λ-trees come under a general construction known as the base change functor. Details may be found in [4], Theorem 4.7, Corollaries 4.8, 4.9.

Theorem 4.1 (Base Change Functor) *Let $h : \Lambda \to \Lambda'$ be an order preserving homomorphism between ordered abelian groups and let G be a group acting by isometries on a Λ-tree, (X, d). Then there is a Λ'-tree, (X', d') on which G acts by isometries and a mapping $\phi : X \to X'$ such that*
(i) $d'(\phi(x), \phi(y)) = h(d(x, y))$, for all $x, y \in X$
(ii) $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X$
(iii) $\|g\|_{X'} = h(\|g\|_X)$ for all $g \in G$.
In particular, if the action of G on X is free and h is injective, then the action of G on X' is also free.

The base change functor by itself will not be enough for our purposes. The next proposition describes another way in which the action on a $\mathbb{Z}^n$-tree may be changed. Although it may seem like a rabbit out of a hat, Proposition 4.2 arises quite naturally when one considers the geometry of a $\mathbb{Z}^n$-tree. To be precise, if one looks at a $\mathbb{Z}^n$-tree X and shrinks the balls of radius $\mathbb{Z}^{n-1}$ to a point, then the resulting object has the structure of a simplicial tree. The $\mathbb{Z}^n$-metric on X is constructed from the metric on the simplicial tree along with what Bass [2] refers to as end maps. These end maps allow a great deal of variation in the metric on X. By adding the same amount to each end map, one can follow the resulting change in the metric. If a group G acts isometrically on X, one can also deduce the change in the translation length function and a similar process can be performed by changing the end maps between the $\mathbb{Z}^k$-balls. Proposition 4.2 reverses this reasoning by starting from a translation length function and producing the $\mathbb{Z}^n$-tree.

Proposition 4.2 *Let (X, d) be a $\mathbb{Z}^n$-tree on which the group G acts by isometries and let $\pi_i : \mathbb{Z}^n \to \mathbb{Z}$ denote the projection onto the i^{th} component of $\mathbb{Z}^n$ (from the left). Then for $1 \leq i \leq n$ and $c \in \mathbb{Z}^{n-i}$ there is a $\mathbb{Z}^n$-tree (X', d') on which G acts isometrically such that $\|g\|' = \|g\| + c\pi_i(\|g\|)$ for all $g \in G$. If the action of G on X is free then so is the action on X'.*

Proof: It is a standard fact (see [14, pp. 297–8] and [1, §8]) that if G acts on a Λ tree X, then the translation function $\|.\| : G \to \Lambda$ satisfies the following axioms.

0. Given $g, h \in G$ with $\|g\| > 0$ and $\|h\| > 0$, then

$$\max\{0, \|gh\| - \|g\| - \|h\|\} \in 2\Lambda$$

I. $\|ghg^{-1}\| = \|g\|$ for every $g, h \in G$.
II. Given $g, h \in G$, either

$$\|gh\| = \|gh^{-1}\| \quad \text{or} \quad \max\{\|gh\|, \|gh^{-1}\|\} \leq \|g\| + \|h\|.$$

III. Given $g, h \in G$ with $\|g\| > 0$ and $\|h\| > 0$, either

$$\|gh\| = \|gh^{-1}\| > \|g\| + \|h\| \quad \text{or} \quad \max\{\|gh\|, \|gh^{-1}\|\} = \|g\| + \|h\|.$$

Conversely, the main theorem of [14] states that given a function $\|.\| : G \to \Lambda$ satisfying the above axioms, there is a Λ tree, X upon which G acts isometrically and with translation length function $\|.\|$. Thus we shall verify that the axioms hold for $\|.\|'$ in the statement of this proposition, given that they already hold for $\|.\|$.

It is clear that since $\|.\|$ satisfies axiom 0, so will $\|.\|'$. Also, if $\|g\| = \|h\|$ then $\|g\|' = \|h\|'$ so that $\|.\|'$ satisfies axiom I. Given that $\|1\|' = 0$, the remaining two axioms may be verified from the following claim.
Claim: Let $g_1, g_2, h_1, h_2 \in G$. If $\|g_1\| + \|g_2\| < \|h_1\| + \|h_2\|$ then $\|g_1\|' + \|g_2\|' < \|h_1\|' + \|h_2\|'$. Also, if $\|g_1\| + \|g_2\| = \|h_1\| + \|h_2\|$ then $\|g_1\|' + \|g_2\|' = \|h_1\|' + \|h_2\|'$.

Proof of claim As $\mathbb{Z}^n$ is ordered lexicographically, $\|g_1\| + \|g_2\| < \|h_1\| + \|h_2\|$ means that for some $1 \leq t \leq n$,

$$\begin{array}{rcl} \pi_j(\|g_1\| + \|g_2\|) & = & \pi_j(\|h_1\| + \|h_2\|), \text{for } 1 \leq j < t, \text{and} \\ \pi_t(\|g_1\| + \|g_2\|) & < & \pi_t(\|h_1\| + \|h_2\|) \end{array}$$

Note that if $k \leq i$ then $\pi_k(\|g_1\|+\|g_2\|) = \pi_k(\|g_1\|'+\|g_2\|')$ and $\pi_k(\|h_1\|+\|h_2\|) = \pi_k(\|h_1\|' + \|h_2\|')$. Hence if $t \leq i$ then $\|g_1\|' + \|g_2\|' < \|h_1\|' + \|h_2\|'$. However, if $t > i$ then $\pi_i(\|g_1\| + \|g_2\|) = \pi_i(\|h_1\| + \|h_2\|)$ which implies that

$$\begin{array}{rcl} \|g_1\|' + \|g_2\|' & = & \|g_1\| + \|g_2\| + c\pi_i(\|g_1\| + \|g_2\|) \\ & < & \|h_1\| + \|h_2\| + c\pi_i(\|h_1\| + \|h_2\|) \\ & = & \|h_1\|' + \|h_2\|'. \end{array}$$

In either case, $\|g_1\|' + \|g_2\|' < \|h_1\|' + \|h_2\|'$.

The second statement of the claim is clear.

With this claim, we can now verify the above axioms for $\|.\|'$. The fact that the resulting action is free follows from the claim since $\|g\| > 0$ for all $g \neq 1$ implies that $\|g\|' > 0$ for all $g \neq 1$.

Our main theorem now follows from the following.

Theorem 4.3 *Let G_1 and G_2 be groups with maximal abelian subgroups, $\langle t_1 \rangle \leq G_1$ and $\langle t_2 \rangle \leq G_2$. Suppose that G_1 acts freely without inversions on a $\mathbb{Z}^{n_1}$-tree X_1 and G_2 acts freely and without inversions on a $\mathbb{Z}^{n_2}$-tree X_2. Then the associated amalgamated free product $G_1 *_{\langle t_1 = t_2 \rangle} G_2$ admits a free action without inversions on a $\mathbb{Z}^n$-tree for some $n \geq \max\{n_1, n_2\}$.*

Note that if $\langle t_i \rangle$ is not maximal abelian in G_i, then $G_1 *_{\langle t_1 = t_2 \rangle} G_2$ fails to be commutative transitive and hence cannot be tree-free.

Proof: We use Theorem 2.4.1(ii) applied to the amalgamated free product, together with the given actions of G_i on X_i modified by several applications of the base change functor and Proposition 4.2 as we now describe. To prevent the notation from becoming too cumbersome, we will abuse notation slightly throughout the proof, and denote a Λ-tree by X_i if it arises from X_i by such modifications. Similarly, we will denote the hyperbolic length functions associated to the action of G_i on X_i by $\|.\|_i$ $(i = 1, 2)$.

Note first that there is no loss of generality in assuming that $n_1 = n_2$, for otherwise, if $n_1 > n_2$ say, we can use the base change functor to replace the $\mathbb{Z}^{n_2}$-tree X_2 by the $\mathbb{Z}^{n_1}$-tree obtained via the embedding

$$(k_1, k_2, \ldots, k_{n_2}) \mapsto (k_1, k_2, \ldots, k_{n_2}, \underbrace{0, 0, \ldots, 0}_{n_1 - n_2}).$$

Hence we may assume that both G_1 and G_2 are $\mathbb{Z}^{n_1}$-free and that the translation length functions have the same codomain, $||.||_i : G_i \to \mathbb{Z}^{n_1}$.

Suppose now that $\|t_i\|_i = m_i e_{j_i} + \lambda_i$, where $m_i > 0$ is an integer, and $\lambda_i \in \mathbb{Z}^{n_1 - j_i}$. Again there is no loss of generality in assuming that $j_1 \geq j_2$ (so that $\|t_1\|_1$ belongs to the convex subgroup of $\mathbb{Z}^{n_1}$ generated by $\|t_2\|_2$). So put $j = j_1 - j_2 \geq 0$, and consider the embeddings

$$h_1 : (k_1, k_2, \ldots, k_{n_1}) \mapsto (k_1, k_2, \ldots, k_{n_1}, \underbrace{0, 0, \ldots, 0}_{j})$$

and

$$h_2 : (k_1, k_2, \ldots, k_{n_1}) \mapsto (\underbrace{0, 0, \ldots, 0}_{j}, k_1, k_2, \ldots, k_{n_1})$$

of $\mathbb{Z}^{n_1}$ in $\mathbb{Z}^n = \mathbb{Z}^{n_1+j}$.

These embeddings enable us to replace the X_i by $\mathbb{Z}^n$-trees on which the G_i act. With respect to these new actions, we have $\|t_1\|_1 = m_1 e_{j_1} + \lambda_1$ and $\|t_2\|_2 = m_2 e_{j_1} + \lambda_2$ (since $j_1 = j + j_2$). Rescaling the metric on X_1 by m_2 and the metric on X_2 by m_1 (and noting that these integers are positive), we now have actions of G_i on X_i where the first non-zero components of $\|t_1\|_1$ and $\|t_2\|_2$ are equal and occur in the same position.

The results of these modifications are actions of G_i on $\mathbb{Z}^n$-trees X_i $(i = 1, 2)$ with $\|t_1\|_1 = me_l + \lambda_1$ and $\|t_2\|_2 = me_l + \lambda_2$ where $m > 0$, $l = j_1$, and $\lambda_i \in \mathbb{Z}^{n-l}$. Applying the base change functor associated to the embedding $h : (k_1, k_2, \ldots, k_n) \mapsto (k_1, k_2, \ldots k_l, mk_{l+1}, \ldots, mk_n)$ to each X_i, we have $\|t_i\|_i = me_l + m\lambda_i$ $(i = 1, 2)$.

Now apply Proposition 4.2 to the action of G_1 on X_1 with $i = l$ and $c = \lambda_2 - \lambda_1$. Then $\|t_1\|_1' = me_l + m\lambda_1 + (\lambda_2 - \lambda_1)m = me_l + m\lambda_2 = \|t_2\|_2$.

The result follows by part (ii) of Theorem 2.4.1.

We note that by a result in [3], Corollary (torsion free products over $\mathbb{Z}$) p. 100, an amalgamated free product $A *_{\mathbb{Z}} B$, where A and B are torsion free hyperbolic groups is itself hyperbolic if $\mathbb{Z}$ is maximal abelian in either A or B. As a tree-free group is necessarily torsion free we may note the following corollary.

Corollary 4.4 *If G_1 and G_2 are hyperbolic then so is G.*

The main reason for noting this corollary is that it allows one to construct many examples of hyperbolic groups which act freely on $\mathbb{Z}^n$-trees. Let $\mathcal{C}$ be the class of hyperbolic groups which act freely on $\mathbb{Z}^n$-trees for some n. Then this class contains all free groups of finite rank and is closed under taking amalgamated free products over an infinite cyclic group, as long as the generator is not a proper power in either of the vertex groups. Thus, $\mathcal{C}$ is a large class of torsion free hyperbolic groups, most of which cannot act freely on an $\mathbb{R}$-tree.

We note that Theorem 4.3 cannot easily be generalised to include the case of HNN-extensions. The reason is that for amalgamated free products it is possible to change the actions of the vertex groups independently so that two given group elements have the same translation length. This is the key ingredient to applying Theorem 4.7 of [2]. To generalise to the case of an HNN-extension, one would also have two group elements which would need to have the same translation length. The fact that both elements would lie in the same group means that one could not ensure this equality by using the base change functor or Proposition 4.2. In fact, it can be shown that the group

$$\langle x, y, t \mid t^{-1}[x, y]t = [x^2, y^2]\rangle$$

cannot act freely on any Λ-tree. This is because for any free action of the free group of rank 2, $F(x, y)$, on a Λ-tree, we have $\|[x, y]\| < \|[x^2, y^2]\|$. Hence the obstruction for HNN-extensions to act freely on $\mathbb{Z}^n$-trees seems to rely more heavily on the group structure.

For completeness we note that certain HNN-extensions, called "benign" in [2], are known to be tree-free. Namely, if G is a Λ-free group and H is a maximal abelian subgroup of G then the HNN-extension

$$\langle G, t \mid t^{-1}ht = h \text{ for all } h \in H\rangle$$

is $(\mathbb{Z} \times \Lambda)$-free by [2], Corollary 4.16.

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